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INTERACTION OF AN UNSTEADY-STATE WAVE WITH A RIGID SPHERE IMMERSED IN A COMPRESSIBLE VISCOUS FLUID*

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The non-stationary interaction of an immovably fixed absolutely rigid sphere immersed in a compressible viscous fluid with a plane unsteady-state longitudinal wave is considered. A Laplace integral transformation is applied to obtain an expression in the transform space for the reaction of the fluid on the sphere. This expression is inverted by analytical and numerical methods. The linearized problems for a compressible viscous fluid considered in this paper produce, in the limit, the standard results for an acoustic medium /1/.

Dynamic problems for rigid bodies immersed in a compressible viscous fluid are considered in the framework of linearized Navier-Stokes equations in /2/. Linearized equations with inertial terms are the equations obtained from the non-linear Navier-Stokes equations as it applies to small-perturbation problems. We have previously used this framework to investigate the non-stationary motion of a rigid sphere under the action of an active force /3/ and an impulse /4/ imparted to the sphere at the initial time. The non-stationary interaction of rigid and deformable bodies with unsteady-state waves propagating in an acoustic medium was considered in particular, in /1, 5/.

1. Statement of the problem and method of solution. Consider a rigid sphere of radius a immersed in an infinite compressible viscous fluid at rest. The fluid motion will be described by linearized Navier-Stokes equations.

The field of the velocity perturbation vector in the fluid can be represented in the form /2/

$$\mathbf{v} = \nabla\Phi + \nabla\times\mathbf{\Psi} \tag{1.1}$$

The scalar potential Φ and the vector potential $\mathbf{\Psi}$ can be obtained from the equations /2/

$$[(1 + {}^4/_3\nu' a_0^{-2}\partial/\partial t)\,\Delta - a_0^{-2}\partial^2/\partial t^2]\,\Phi = 0 \tag{1.2}$$

$$(\nu'\Delta - \partial/\partial t)\,\mathbf{\Psi} = 0 \tag{1.3}$$

where a_0 is the velocity of propagation of small perturbations in the fluid, and ν' is the kinematic coefficient of viscosity.

The pressure and density perturbations P and ρ can be represented in the form (ρ_0 is the fluid density)

$$P = \rho_0\,({}^4/_3\nu'\Delta - \partial/\partial t)\,\Phi \tag{1.4}$$

$$\rho = \rho_0 a_0^{-2}\,({}^4/_3\nu'\Delta - \partial/\partial t)\,\Phi \tag{1.5}$$

We assume that the fluid is initially at rest,

$$t = 0,\ u = v = P = 0 \tag{1.6}$$

(u is the displacement of the fluid particles and v is their velocity).

We associate Cartesian (x, y, z) and spherical (r, θ, φ) coordinate systems with the sphere, measuring the angle θ from the z-axis. Assume that the sphere is fixed immovably. Let us determine the loads acting on the sphere as it interacts with a plane unsteady-state pressure wave propagating in the negative direction of the z-axis.

The initial pressure perturbation P in the fluid in the plane $z = l + a$ is given by

$$P = f(t)\,H(t) \tag{1.7}$$

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Here $f(t)$ is a function and $H(t)$ is the Heaviside function. Let Φ° be the incident wave potential and Φ^* and Ψ^* the reflected wave potentials. By convention, all variables relating to the incident wave will be given the superscript $^\circ$ and all variables relating to reflected waves will be given the superscript *. These potentials satisfy Eqs.(1.2) and (1.3), while the pressure perturbations P° in the incident wave and P^* in the reflected waves satisfy (1.4).

The boundary conditions for $t > 0$ have the form

$$r = a,\ v_r^\circ + v_r^* = 0,\ v_\theta^\circ + v_\theta^* = 0 \tag{1.8}$$

The problem thus reduces to the following: simultaneously solve Eq.(1.2) for the incident wave potential Φ°, and Eqs.(1.2) and (1.3) for the reflected wave potentials, satisfying the zero initial conditions

$$t = 0,\ \Phi^\circ = \Phi^* = 0,\ \Psi^* = 0 \tag{1.9}$$

and the boundary conditions (1.8).

Given the zero initial conditions, we apply a Laplace integral transformation to Eqs. (1.2) and (1.3), and also to the boundary conditions (1.8), and solve the problems in the transform domain.

2. *Determination of the reaction of the fluid on the sphere in the transform space.* The suggested approach leads to the following problem in the transform space (p is the transformation parameter):

$$(\partial^2/\partial z^2 - s^2)\,\Phi^{\circ L} = 0,\ (\Delta - s^2)\,\Phi^{*L} = 0,\ (\nu'\Delta - p)\,\Psi^{*L} = 0 \tag{2.1}$$

$$P^{\circ L} = \rho_0\,({}^4/_3\nu'\partial^2/\partial z^2 - p)\,\Phi^{\circ L},\quad P^{*L} = \rho_0\,({}^4/_3\nu'\Delta - p)\,\Phi^{*L},\quad P^{\circ L}|_{z=l+a} = f^L(p)$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} \tag{2.2}$$

$$s = \frac{p}{a_0\sqrt{1 + {}^4/_3\nu' a_0^{-2}p}}$$

$$r = a,\quad v_r^{\circ L} + v_r^{*L} = 0,\quad v_\theta^{\circ L} + v_\theta^{*L} = 0 \tag{2.3}$$

The potential of a plane wave propagating along the z-axis satisfies the first equation in (2.1). Its general solution has the form

$$\Phi^{\circ L} = A\,(p)\,e^{-sz} + B\,(p)\,e^{sz} \tag{2.4}$$

For a wave propagating in the negative direction of the z-axis, $A\,(p) = 0$. The coefficient $B\,(p)$ is determined from the first and the last relationships in (2.2). As a result, we obtain an expression for the potential of the incident wave, and also the pressure created by the incident wave in the compressible viscous fluid:

$$\Phi^{\circ L} = -\frac{1 + {}^4/_3\nu' a_0^{-2}p}{\rho_0 p}\,f^L(p)\,e^{-s(l+a-z)},\quad P^{\circ L} = f^L(p)e^{-s(l+a-z)} \tag{2.5}$$

We can show that the pressure in the compressible viscous fluid created by the pressure jump in the plane $z = l + a$ is non-zero everyhwere in the medium for $t > 0$. Indeed, taking an asymptotic limit of the last expression in (2.5) for large $|p|$ and inverting it, we obtain for $f(t) = 1$

$$P(z, t) = 1 - \operatorname{erf}\left(\frac{l + a - z}{4}\frac{\sqrt{3}}{\sqrt{\nu' t}}\right) \tag{2.6}$$

This necessitates introducing a new parameter l - the distance from the perturbation source to the body - to analyse the propagation of unsteady-state waves in a compressible viscous fluid, with the time measured from the instant when the perturbation is injected into the fluid. Passage to the limit in an acoustic medium obviously involves the substitution $t \to t + l/a_0$ as $\nu' \to 0$.

For the axisymmetric case of reflection from the sphere, the vector potential Ψ is given by /2/

$$\Psi = \nabla\times(r e_r \Psi_2) \tag{2.7}$$

The solutions of the second and the third equations in (2.1) allowing for the decay of perturbations at infinity have the form

$$\Phi^{*L} = \sum_{n=0}^{\infty} C_n\,(p)\,r^{-1/2}K_{n+1/2}\,(sr)P_n\,(\cos\theta) \tag{2.8}$$

$$\Psi_2^{*L} = \sum_{n=0}^{\infty} D_n(p)\, r^{-1/2} K_{n+1/2}(qr)\, P_n(\cos\theta), \quad q = \sqrt{\frac{p}{\nu'}}$$

The coefficients $C_n(p)$ and $D_n(p)$ will be determined by satisfying the boundary conditions (2.3). To this end, we expand the incident wave potential $\Phi^{\circ L}$ in a Legendre polynomial series, noting that $z = r\cos\theta$:

$$\Phi^{\circ L} = -\frac{1 + {}^4/_3\nu' a_0^{-2} p}{\rho_0 p} f^L(p)\, e^{-s(l+a)} \sqrt{\frac{\pi}{2sr}} \sum_{n=0}^{\infty} (2n+1)\, I_{n+1/2}(sr)\, P_n(\cos\theta) \tag{2.9}$$

Expressing the components of the fluid velocity vector in terms of the potentials $\Phi^{\circ L}$, Φ^{*L}, and Ψ_2^{*L} and substituting them into (2.3), we obtain a system of two algebraic equations for determining $C_n(p)$ and $D_n(p)$ for all n. These equations are quite complicated and are therefore omitted.

The stress vector on a surface element with the normal $\mathbf{N}$ in the fluid is determined by the components of the stress tensor in the fluid

$$\mathbf{T}_N = \mathbf{e}_r\sigma_{rr} + \mathbf{e}_\theta\sigma_{r\theta} \tag{2.10}$$

The stress vector on the surface of the sphere in the fluid $\mathbf{T}_N = \mathbf{T}_N^\circ + \mathbf{T}_N^*$ is epxressed in terms of the stress tensor components σ_{rr}°, $\sigma_{r\theta}^\circ$, σ_{rr}^*, $\sigma_{r\theta}^*$. These components in turn are expressed in terms of Φ°, Φ^*, and Ψ_2^*, respectively /2/.

The stress tensor components in the transform space have the form ($\mu' = \rho_0\nu'$)

$$\sigma_{rr}^{\circ L} = 2\mu'\left(\frac{\rho_0 p}{2\mu'} - \Delta + \frac{\partial^2}{\partial r^2}\right)\Phi^{\circ L}, \quad \sigma_{r\theta}^{\circ L} = 2\mu'\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\Phi^{\circ L} \tag{2.11}$$

$$\sigma_{rr}^{*L} = 2\mu'\left[\left(\frac{\rho_0 p}{2\mu'} - \Delta + \frac{\partial^2}{\partial r^2}\right)\Phi^{*L} + \left(r\frac{\partial^3}{\partial r^3} + 3\frac{\partial^2}{\partial r^2} - r\frac{\partial}{\partial r}\Delta - \Delta\right)\Psi_2^{*L}\right]$$

$$\sigma_{r\theta}^{*L} = 2\mu'\left[\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\Phi^{*L} + \frac{\partial}{\partial\theta}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{\Delta}{2}\right)\Psi_2^{*L}\right]$$

The component (along the $\mathbf{e}_3$ unit vector) of the reaction of the fluid on the sphere is given by the expression

$$F_3^L = 2\pi a^2 \int_0^\pi (\mathbf{T}_N\cdot\mathbf{e}_3)|_{r=a}\sin\theta\, d\theta = \tag{2.12}$$

$$2\pi a^2 \int_0^\pi \{\cos\theta\,(\sigma_{rr}^{\circ L} + \sigma_{rr}^{*L}) - \sin\theta\,(\sigma_{r\theta}^{\circ L} + \sigma_{r\theta}^{*L})\}\sin\theta\, d\theta$$

Integrating over the surface of the sphere and using the orthogonality of Legendre polynomials and the fact that $K_{n+1/2}(z)$ and $I_{n+1/2}(z)$ are expressible in terms of elementary functions, we obtain after some reduction an expression in the transform space for the reaction of the fluid on a sphere interacting with a plane unsteady-state wave:

$$F_3^L(p) = -4\pi a^2\left(1 + {}^4/_3\nu' a_0^{-2} p\right) f^L(p)\, e^{-\lambda\beta} Q^L(p) / (1 + K^L(p)) \tag{2.13}$$
$$Q^L(p) = \beta^{-1} + 3\beta^{-1}\gamma^{-1} + 3\beta^{-1}\gamma^{-2}, \quad K^L(p) = 2\beta^{-1} + 2\beta^{-2} + \gamma^{-1} + \gamma^{-2}$$

Here $\beta = sa$, $\gamma = qa$, and $\lambda = l/a$.

In order to investigate (2.13), we will change to a dimensionless transformation parameter $\bar{p} = ap/a_0$, corresponding to the dimensionless time $\tau = a_0 t/a$ in the source space. In dimensionless variables,

$$\beta = \bar{p}\,(1 + \bar{p}/k)^{-1/2}, \quad \gamma = 2\,(k\bar{p}/3)^{1/2}, \quad k = {}^3/_4 a a_0/\nu'$$

In what follows, we simply write p for $\bar{p}$.

3. *Limiting cases.* Passage to the limit $\nu' \to 0$ in the acoustic medium reduces (2.13) to the form

$$F_{3ak}^L(p) = -4\pi a^2 f^L(p)\, e^{-\lambda p} p\,(p^2 + 2p + 2)^{-1} \tag{3.1}$$

For the case $f(\tau) = 1$, the inverse of (3.1) has the form

$$F_{3ak}(\tau) = -4\pi a^2 e^{-(\tau-\lambda)}\sin(\tau - \lambda)\, H(\tau - \lambda) \tag{3.2}$$

which is identical with the result obtained for the acoustic medium model /1/ after making the substitution $\tau \to \tau + \lambda$.

Let us investigate the variation of the reaction of a compressible viscous fluid on a sphere for large τ. To this end, expression (2.13) is replaced with an asymptotically equivalent expression for small $|p|$. In the function $Q^L(p)$ we retain the second and the third terms and in the denominator we retain $2\beta^{-2}$. The exponential is set equal to 1. As a result, we obtain

$$F_3^L(p) \approx -6\pi a^2 f^L(p)(\beta\gamma^{-1} + \beta\gamma^{-2}) \tag{3.3}$$

For the case $f(\tau) = P_0$ ($P_0 = \text{const}$), the inverse of (3.3) has the form /6/

$$F_{3ac}(\tau) = -6\pi a^2 P_0 ({}^1/_2 \sqrt{3} e^{-u} I_0(u) + {}^3/_4 k^{-1} \operatorname{erf} \sqrt{2u}), \quad u = {}^1/_2 k\tau \tag{3.4}$$

In the limit as $\tau \to \infty$ we obtain from (3.4)

$$F_3(\infty) = -6\pi a v' a_0^{-1} P_0 \tag{3.5}$$

From (3.5) we obtain the classical Stokes formula for fluid flow past a sphere. Indeed, consider the limiting value of the velocity of the fluid particles, as it follows from the expression for the potential $\Phi^{\circ L}$ (2.5) and from the limit theorems of operational calculus

$$v_z(\infty) = \partial\Phi^\circ/\partial z_{|t\to\infty} = \lim_{p\to 0} p\partial\Phi^{\circ L}/\partial z = P_0/(\rho_0 a_0) \tag{3.6}$$

Thus, the limiting reaction of the fluid on a fixed sphere as $\tau \to \infty$ in the case $f(\tau) = P_0$ is identical with the standard Stokes formula.

4. ***Numerical determination of the reaction of the fluid on the sphere.*** In order to invert expression (2.13), we represent it in the form

$$F_3^L(p) = \exp[-\lambda p(1 + p/k)^{-1/2}] Y^L(p) \tag{4.1}$$

The function

$$Y^L(p) = \frac{4\pi a^2 (1 + p/k) f^L(p) Q^L(p)}{1 + K^L(p)} \tag{4.2}$$

in the source space corresponds to a Volterra linear integral equation of the second kind /7/

$$Y(\tau) + \int_0^\tau K(\tau - s) Y(s)\, ds = G(\tau) \tag{4.3}$$

Here $G(\tau)$ is the source of the numerator in formula (4.2).

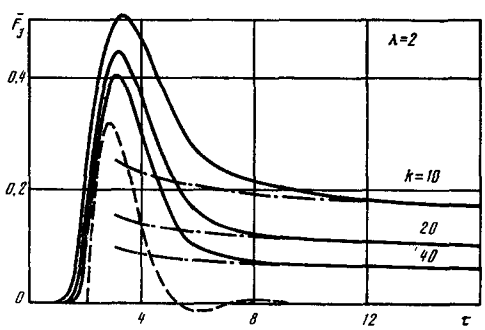

Fig.1

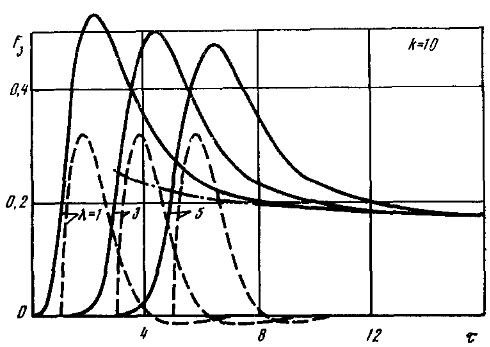

Fig.2.

Eq.(4.3) was solved numerically by successive approximations. The source of $\exp[-\lambda p(1 + p/k)^{-1/2}]$ is determined by representing this function in series form and inverting the series term by term. The sum of the series is determined numerically. Then the reaction of the fluid on the sphere $F_3(\tau)$ is obtained numerically by convolution.

Figs.1 and 2 plot the reaction of the medium on the sphere $\bar{F}_3 = -F_3/(4\pi a^2)$ as a function of the dimensionless time τ for $f(\tau) = 1$ and various values of the parameters k (characterizing the size of the sphere and the properties of the fluid) and λ (the dimensionless distance from the perturbation source to the sphere). The solid curves correspond to the case of a compressible viscous fluid, and the dashed curves represent the solution (3.2) for the acoustic medium. The dash-dot curves correspond to the asymptotic approximation of the fluid reaction (3.4).

Unlike the acoustic medium, where the reaction on the sphere is produced by the incident wave front for $\tau = \lambda$, the reaction on the sphere in a compressible viscous fluid is non-zero for $\tau > 0$. For $\lambda = 2$, as the viscosity increases (Fig.1), the reaction of the compressible viscous fluid on the sphere progressively deviates from the reaction of the acoustic medium. The amplitude of the reaction of the fluid on the sphere noticeable decreases with distance from the perturbation source (Fig.2) and becomes smoother. The numerical results agree closely with the asymptotic approximation (3.4) for moderate dimensionless times τ.

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ON THE IMPOSSIBILITY OF REGULAR REFLECTION OF A STEADY-STATE SHOCK WAVE FROM THE AXIS OF SYMMETRY*

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Some problems concerning the explanation of the fact that regular reflection of a shock wave from the axis of symmetry is impossible are considered. This fact is well-known and can be demonstrated by linear analysis; it was proved in /1/ by integrating the compatibility condition along the characteristic reaching the point of alleged regular reflection. In this paper, we investigate the flow in the neighbourhood of this point and show that it should be conical. We also prove that the inverse problem of constructing the flow field and the boundary streamline from a given shock wave of arbitrary shape is physically unrealizable in a small neighbourhood of the axis of symmetry.

This topic is also relevant because the literature contains conflicting statements claiming that regular reflection is possible and (much more seldom) impossible, never offering a detailed explanation (see, e.g., /2, 3/). This may explains why this topic has not been treated in detail in authoritative monographs, unlike the similar problem of the collapse of an unsteady-state spherical or cylindrical shock wave.

1. Consider the following proposed picture of the supersonic axisymmetric flow of an

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